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Monotone Dynamics, I

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Why study monotone systems?

> Monotone systems respond in a predictable fashion to perturbations, and have very robust dynamical characteristics. This makes them reliable components of more complex networks, and suggests that natural biological systems may have evolved to be, if not monotone, at least close to monotone. In addition, interconnections of monotone systems may be fruitfully analyzed using tools from control theory.
>
> —E. D. Sontag 2007

ODEs

Consider a vector differential equation on a set $X \subset \mathbb{R}^n$:

$$\dot{x} = F(x), \qquad (t \geq 0,\ x \in X)$$

X is connected, its interior is dense in X, and $F\colon\ X \to \mathbb{R}^n$ is C^1 (continuously differentiable).

The solution with initial value $y \in X$ is denoted by $t \mapsto \Phi_t y$.

Φ_t is a homeomorphism between subsets of X.
Φ_0 is the identity map of X, and $\Phi_s \circ \Phi_t = \Phi_{s+t}$.

The set $\Phi := \{\Phi_t\}_{t \geq 0}$ is the **local semiflow** generated by F.

Local semiflows

Now let Φ be any local semiflow in a metric space.

The **equilibrium set** $\mathcal{E}$ is the set of points fixed under every Φ_t.
If Φ comes from a vector field F then $\mathcal{E}(F) := \mathcal{E} = F^{-1}(0)$.

$\bar{O}(x)$ denotes the closure of the **orbit**

$$O(x) := \{\Phi_t x\colon\ t \geq 0\}$$

The **omega limit set** $\omega(x)$ is the set of limit points of all sequences $\Phi_{t_k} x,\ t_k \to \infty$.

A **cycle** is a periodic orbit that is not an equilibrium.

x is **convergent** if $\Phi_t x$ converges (necessarily to an equilibrium).

$A \subset X$ is **positively invariant** if $\Phi_t A \subset A$, and **invariant** if A is nonempty and $\Phi_t A = A$, for all $t \in [0, \infty)$.

Ordered spaces

An **ordered space** Y is a metric space with a closed order relation $\leq$. We write

$$\begin{aligned} x < y &\iff x \leq y \text{ and } x \neq y, \\ < \textit{all} > x \ll y &\iff U < V \text{ for respective neighborhoods } U, V \text{ of } x, y \\ < \textit{all} > y \geq x &\iff x \leq y, \quad \text{etc.} \end{aligned}$$

Usually $Y \subset \mathbb{R}^n$ with the **vector order**:

$$x \leq y \iff x_i \leq y_i, \quad x \ll y \iff x_i \leq y_i \qquad (i = 1, \ldots, n)$$

All maps are assumed continuous.
A map f between ordered spaces is called

monotone if $x \leq y \implies f(x) \leq f(y)$,
strongly monotone if $x < y \implies f(x) \ll f(y)$.

Monotone local semiflows

Let Φ be a local semiflow in an ordered space Y.
Call Φ

monotone if Φ_t is monotone for $t \geq 0$,
strongly monotone if Φ_t is strongly monotone for $t > 0$.

The convergence criterion:
Assume Φ is monotone and $y \in Y$ has compact orbit closure. Then y is convergent provided there is an interval $J \subset \mathbb{R}_+$ such that either

$$t \in J \implies \Phi_t y \leq y, \quad \textit{or}$$
$$t \in J \implies \Phi_t y \geq y.$$

If Φ is strongly *monotone, y is convergent provided:*

$$\Phi_s y < y \textit{ or } \Phi_s y > y \textit{ for some } s \in \mathbb{R}_+$$

Cooperative vector fields

A C^1 vector field F in X is

cooperative if $\dfrac{\partial F_i}{\partial x_j} \geq 0, \quad (i \neq j)$,

irreducible if the matrices $\left[\dfrac{\partial F_i}{\partial x_j}(p)\right]$ are irreducible, $(p \in X)$

Theorem 1. *Assume X is convex. A system $\dot{x} = F(x)$ is:*

- *monotone if F is cooperative* (—Müller 1926, Kamke 1932)
- *strongly monotone if F is cooperative and irreducible* (—Hirsch 1985)

Examples in the positive orthant $\mathbb{R}^n_+ := [0,\infty)^n$

Let A be an $n \times n$ matrix with $A_{ij} \geq 0$ for $i \neq j$.

Lotka-Volterra systems:
The system in $\mathbb{R}^n_+$,

$$\dot{x}_i = x_i\left[b_i + \sum_j A_{ij}x_j\right], \quad A_{ij} \geq 0 \qquad (i \neq j)$$

is cooperative and hence monotone, and strongly monotone if also A is irreducible.

Similarly for

Excitatory neural network models:

$$\dot{x}_i = x_i\sigma_i\big(b_i + \sum_j A_{ij}x_j\big), \quad {\sigma_i}' > 0$$

Attractors

Let Φ be a local semiflow in X. A nonempty set $A \subset X$ **attracts** $Z \subset X$ if:

$$\lim_{t\to\infty} \mathsf{dist}(\Phi_t x, A) = 0 \text{ uniformly for } x \in Z$$

A is an **attractor**, or **attracting**, if:

A is compact and invariant, and attracts a neighborhood of A

The union of such neighborhoods is a positively invariant open set $\mathcal{B}$, the **basin** of A. If $\mathcal{B} = X$ then A is the **global attractor**.

$p \in \mathcal{E}$ is **asymptotically stable** if it is an attractor, and **global asymptotically stable** (=GAS) if p is the global attractor.

A compact invariant set S is **stable** if every neighborhood of S contains an invariant neighborhood of S. Otherwise S is **unstable**.

On the concept of "Attractor"

Attractors do not occur explicitly in the work of Poincaré or Birkhoff. They were primarily interested in Hamiltonian systems, which have no attractors because they preserve volume.

In 1952 Turing published pictures of computer simulations of a nonlinear dynamical model of cell development, exhibiting striking pattern formation.

E. Lorenz (1963), and P. Stein & S. Ulam (1959, 1964), published pictorial evidence of complicated structure in attractors.

But R. Hamming's review of Stein & Ulam was unenthusiastic:

R. Hamming, 1965

Review of Stein and Ulam:
Many photographs of cathode ray tube displays are given, a fondness for citing large numbers of iterations and machine time used is revealed, and a crude classification of the limited results is offered, but there appears to be no firm new results of general mathematical interest...

One can only wonder what will happen to mathematics if we allow the undigested outputs of computers to fill our literature. The present paper shows only slight traces of any digestion of the computer output.

Attractors and chaos

F. Takens & D. Ruelle (1971) showed that "strange attractors" could arise from loss of stability in successive Hopf-type bifurcations:

fixed point $\rightarrow$ periodic orbit $\rightarrow$ invariant torus $\rightarrow \cdots$

They proposed this scenario as a model of the onset of turbulence in fluid dynamics. The physical significance of this route to chaos is still being debated.

Attractors and morphogenesis

In his controversial book "Morphogenesis," R. Thom proclaimed a fundamental scientific role for attractors:

René Thom, 1972

1. Every object, or every physical form, can be represented by an *attractor* C of a dynamical system in a space M of *internal variables.*

2. Such an object possesses no stability, and for this reason cannot be perceived, unless the corresponding attractor is *structurally stable.*

3. Every creation or destruction of forms, every morphogenesis, can be described by the disappearance of the attractors representing the initial forms and their replacement through capture by the attractors representing the final forms. This process, called *catastrophe*, can be described in a space of *external variables*. ...

What good is monotonicity?

Attractors are not chaotic

When Φ is monotone in X the following hold:

- *Every orbit is nowhere dense.*
- *If X is open in $\mathbb{R}^n$ or $\mathbb{R}^n_+$ and A is a nonequilibrium attractor, no orbit is dense in A,*
- *There are no attracting cycles.*
- *If $X = \mathbb{R}^n$ or $\mathbb{R}^n_+$ and A is the global attractor:*
  - $\sup A$ *and* $\inf A$ *exist and are equilibria,*
  - *if $\mathcal{E} = p$ then p is globally asymptotically stable.*

But notice the restrictions on the domain X.

Chain recurrence

Let Φ be a local semiflow in X with a global attractor. The set of **chain recurrent** points is nonempty, compact and invariant. It contains all omega limit points, recurrent points, nonwandering points, heteroclinic and homoclinic orbits.

Chain recurrence in a vector field F:
p is chain recurrent $\iff$ F can be pointwise *approximated by a vector field having a periodic orbit through p.*

Strict Liapunov functions

A **strict liapunov function** (=SLF) for Φ is a map $V\colon X \to [0, \infty)$ such that:

$$x \notin \mathcal{E} \implies V(\Phi_t x) < V(x)$$

C. Conley's Theorem

> *There exists an SLF $\iff$ every chain recurrent point is an equilibrium.*

When Φ has an SLF:

- There are no cycles, no heteroclinic or homoclinic orbits, no recurrent orbits except equilibria.

- If Φ come from a vector field F and G is a sufficiently pointwise close to F, all omega limit points for G lie in any given neighborhood of $\mathcal{E}(F)$.

Convergence in cooperative Lotka-Volterra system

Theorem 2. *Consider a system in* $\mathbb{R}^n_+$*:*

$$\dot{x}_i = x_i \left[b_i + \sum_j A_{ij} x_j \right], \quad b_i > 0, \quad A_{ij} \geq 0 \ \ \text{if} \ \ i \neq j$$

and assume a global attractor. Then:

- *every open face* $L \subset \mathbb{R}^n_+$ *contains a unique equilibrium* p_L
- p_L *is GAS for* $\Phi|L$
- *every chain recurrent point is an equilibrium (hence every trajectory converges)*
- *there is a strict liapunov function*

Limit sets in strongly monotone local semiflows

The Limit Set Dichotomy
Assume Φ *is strongly monotone,* $x < y$*, and* $\bar{O}(x)$*,* $\bar{O}(y)$ *are compact. Then one of the following holds:*

- $\omega(x) \ll \omega(y)$
- $\omega(x) = \omega(y) = p \in \mathcal{E}$.

—M.W. Hirsch; H.L. Smith & H.Thieme

On the other hand:

Monotonicity is not enough:
There is a cooperative vector field in $\mathbb{R}^2$ for which the Limit Set Dichotomy fails.

—E. Sontag & Y. Wang

Generic convergence in strongly monotone systems

$\mathcal{C}$ denotes the set of convergent points for a strongly monotone local semiflow.

Theorem 3. *Assume Φ is strongly monotone with compact orbit closures. Then:*

- *$\mathcal{C}$ contains a dense open set,*
- *$X \setminus \mathcal{C}$ has measure zero,*
- *for every totally ordered arc $J \subset X$, the set $J \setminus \mathcal{C}$ is countable and nowhere dense,*
- *every cycle is unstable.*

—Hirsch; Smith & Thieme; Enciso *et al.*

Conjecture:
This holds even if Φ is merely monotone.

Existence of stable equilibria

Theorem 4. *Assume:*

- *$X \subset \mathbb{R}^n_+$ is an open neighborhood of* 0
- *Φ is strongly monotone*
- *$\mathcal{E}$ is compact*

Then:

(a) *there is a global attractor,*

(b) *every attractor A contains a stable equilibrium p. If p is unique then $A = p$.*

(c) *if $\mathcal{E}$ is finite or F is analytic, every attractor contains an asymptotically stable equilibrium,*

(d) *if $\#\mathcal{E} \leq 2$, every bounded trajectory converges.*

Embedding arbitrary systems in strongly monotone systems

G. A. Enciso has recently proved the following result, generalizing a result of Smale for competitive systems:

Theorem 5. *Every C^1 vector field in a compact subset of $\mathbb{R}^{n-1}$ embeds as an invariant set in some cooperative irreducible vector field F in $\mathbb{R}^n$.*

This shows that the *abstract* dynamics of cooperative irreducible systems can be quite complex. For example:

Chaos in strongly monotone systems:
There are cooperative, irreducible systems in $\mathbb{R}^4_+$ having Smale horseshoes as invariant subsets.

But these cannot be attractors!

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