Monotone Dynamics, II

Morris W. Hirsch

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Competitive systems

A system $\dot{x} = F(x)$ in is **competitive** if

$$\frac{\partial F_i}{\partial x_j} \leq 0 \ \text{ for } i \neq j,$$

Equivalently: the **time-reversed system** $\dot{y} = -F(y)$ is cooperative.

The long-term dynamics of general competitive systems have few of the nice properties of cooperative systems— there can be attractors that are cycles, or even chaotic. But some some results for cooperative systems imply nice things for competitive systems, because:

- *F and $-F$ have the same compact invariant sets.*

Invariant hypersurfaces

Let F be a C^1 vector field in an open subset X of $\mathbb{R}^n$ or $\mathbb{R}^n_+$.

Theorem 1. *Assume F is competitive or cooperative and $L \subset X$ is a compact ω-limit set. Then $L \subset \Sigma \subset X$, such that:*

- *Σ is a positively invariant Lipschitz hypersurface,*
- *no points of Σ are related by $\ll$,*
- *no points of Σ are related by $<$ provided F is irreducible.*

When $n = 3$ then Σ embeds in $\mathbb{R}^2$. The Poincaré-Bendixson Theorem applies to the dynamics in Σ, therefore:

Corollary 2. *If $n = 3$ and L does not contain an equilibrium, then L is a cycle.*

Cascades

Joint work with D. Angeli & E. D. Sontag
Assume the interiors of $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^k$ are open in X and Y respectively.

A **cascade** (of length 1) in $X \times Y \subset \mathbb{R}^m \times \mathbb{R}^k$ is a system of the form

$$\dot{x} = G(x), \quad \dot{y} = H(x, y), \qquad (x, y) \in X \times Y$$

where

$$G\colon\ X \to \mathbb{R}^n, \quad H\colon\ X \times Y \to \mathbb{R}^k$$

This system corresponds to the vector field

$$F\colon\ X \times Y \to \mathbb{R}^m \times \mathbb{R}^k, \quad F(x, y) = (G(x), H(x, y))$$

We say F is a **cascade over G**, denoted by $F \twoheadrightarrow G$.

Total, base and fibre systems

F and G induce respective local semiflows Φ in $X \times Y$ and Ψ in X. The projection $\boldsymbol{\pi}\colon\ X \times Y \to X$ sends $F(x, y)$ to $G(x)$, and takes trajectories of the **total system** F to trajectories of the **base system** G:

$$\boldsymbol{\pi} \circ \Phi_t = \Psi_t \circ \boldsymbol{\pi}$$

Suppose $p \in \mathcal{E}(G)$. Then p is fixed under Ψ, hence the **fibre** $p \times Y$ is positively invariant under Φ. The natural identification $p \times Y \approx Y$ conjugates $\Phi|(p \times Y)$ to the local semiflow of the **fibre system over** p:

$$\dot{y} = H(p, y), \qquad y \in Y$$

Derivatives in cascades

Let $F \twoheadrightarrow G$ be a cascade. The derivative of F has the block form

$$F' = \begin{bmatrix} \frac{\partial G}{\partial x} & 0 \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{bmatrix}$$

Conversely: A vector field F in $X \times Y$ is a cascade over some vector field G in Y provided the X-coordinates of $F(x, y)$ are independent of y.

Even if the base and all fibre systems are cooperative, the total system F need not be cooperative.

But certain good properties of *are* inherited by F.

Inherited properties in cascades

Theorem 3. *Each of the following properties holds for the total system of a cascade* $\Longleftrightarrow$ *it holds for the base system and all fibre systems:*

- *there is a global attractor*
- *every orbit is nowhere dense*
- *orbit closures are compact,* $\mathcal{E}$ *is finite, and there is a strict Liapunov function*

Convergence in quasicooperative Lotka-Volterra system

Consider a system in $\mathbb{R}^n_+$:

$$\dot{x}_i = x_i\left[b_i + \sum_j A_{ij}x_j\right], \quad t \geq 0$$

Theorem 4. *Assume:* $b_i > 0$*, matrix* $[A_{ij}]$ *is in lower triangular block form with diagonal blocks having nonnegative off-diagonal entries, and there is a global attractor. Then every solution converges. Moreover:*

- *Every open face* L *of* $\mathbb{R}^n_+$ *contains a unique equilibrium* p_L*, globally asymptotically stable for* $\Phi|L$*.*
- *There is a strict liapunov function.*

Proof: Induction on the number of diagonal blocks: For one block the system is cooperative and the result known. The inductive step follows from inheritance properties of cascades.

Iterated cascades

Set $n = n_0 + \cdots + n_\nu$. A **cascade of length** ν and **type** $(n_0, \ldots, n_\nu)$ is a system

$$\boxed{\dot{x} = F(x), \quad x \in X^0 \times \cdots \times X^\nu \ \subset\ \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_\nu} = \mathbb{R}^n}$$

expressed as

$$\begin{aligned} \dot{x}^0 &= G(x^0) \\ \dot{x}^1 &= H^1(x^0, x^1) \\ &\vdots \\ \dot{x}^\nu &= H^\nu(x^0, \ldots, x^\nu) \end{aligned}$$

The first $\mu+1$ rows of this system form a system F^μ which is a cascade of length μ. Thus F decomposes into a sequence of ν cascades:

$$\boxed{F = F^\nu \twoheadrightarrow \cdots \twoheadrightarrow F^1 \twoheadrightarrow G}$$

Quasicooperative systems

F has a cascade decomposition of type $(n_0, \ldots, n_\nu)$
$\iff$ each matrix $F'(x)$ has the triangular block form

$$\begin{bmatrix} B_{00} & 0 & \ldots & & & 0 \\ B_{10} & B_{11} & 0 & \ldots & & 0 \\ B_{20} & B_{21} & B_{22} & 0 & \ldots & 0 \\ \vdots & & & & \ddots & \vdots \\ & & & & & 0 \\ B_{\nu 0} & B_{\nu 1} & \ldots & & B_{\nu,\nu-1} & B_{\nu\nu} \end{bmatrix}$$

where matrix $B_{kl} = B_{kl}(x_0, \ldots, x_l)$ has n_k columns and n_l rows.

Call F is **quasicooperative** if the diagonal blocks B_{kk} have nonnegative off-diagonal entries.

A quasicooperative system is a feed-forward net of cooperative systems

For each a in a finite ordered set $\mathcal{A}$, consider a control system

$$\dot{x}^a = F^a(x^a; \xi^a) \qquad (\boldsymbol{S}^a)$$

such that:

$$\text{System } (\boldsymbol{S}^a) \text{ is cooperative for each } \xi^a \qquad (\boldsymbol{C}^a)$$

Suppose each parameter vector ξ^a is specified as a function of the vectors x^b, $(b < a)$. Concatenating the systems $(\boldsymbol{S}^a)$ determines a quasicooperative system.

Conversely: every quasicooperative system can be represented as a feedforward net of control systems $(\boldsymbol{S}^a)$ satisfying condition $(\boldsymbol{C}^a)$.

Transforming systems to cooperative of quasicooperative

Some systems can be made cooperative by changing the signs of some variables: The competitive system in $\mathbb{R}^2$

$$\dot{x} = -y, \quad \dot{y} = -x$$

becomes the cooperative system $\dot{x} = u, \quad \dot{u} = x$ by setting $y = -u$. On the other hand, the system in $\mathbb{R}^3$

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= x + y - z \\ \dot{z} &= -x - y - z, \end{aligned}$$

with Jacobian matrices $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$, cannot be made cooperative by changing signs of variables, but setting $y = -u$ make it quasicooperative.

The interaction graph

Let F be a vector field in $X \subset \mathbb{R}^n$.

The directed, labeled **interaction graph** $\Gamma(F)$ has vertices $1, \dots, n$.

There is a directed edge (j, i) provided $\dfrac{\partial F_i}{\partial x_j}$ is not identically zero.

Edges are assigned labels in $\{+1, -1, \theta\}$ according to the

Labeling rule:

$$h(j,i) = \begin{cases} +1 & \text{if } \dfrac{\partial F_i}{\partial x_j} \geq 0 \quad \text{(a \textbf{positive edge})} \\ -1 & \text{if } \dfrac{\partial F_i}{\partial x_j} \leq 0 \quad \text{(a \textbf{negative edge})} \\ \theta & \text{otherwise} \quad \text{(an \textbf{ambiguous edge})} \end{cases}$$

Loops in the interaction graph

A **path** $\boldsymbol{\lambda}$ of length $k \geq 1$ is a sequence of edges $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$ with

$$\lambda_l = (u_{l-1}, u_l), \qquad (l = 1, \dots, k)$$

The path $\boldsymbol{\lambda}$ is a **loop** if $k \geq 2$ and $u_k = u_0$.

$\boldsymbol{\lambda}$ is given the label

$$h(\boldsymbol{\lambda}) = \begin{cases} h(\lambda_1) \cdots h(\lambda_l) \in \{\pm 1\} & \text{if no edge } \lambda_l \text{ is ambiguous,} \\ \theta & \text{otherwise} \end{cases}$$

$$\boldsymbol{\lambda} \text{ is } \begin{cases} \textbf{positive} & \text{if } h(\boldsymbol{\lambda}) = 1 \\ \textbf{negative} & \text{if } h(\boldsymbol{\lambda}) = -1 \end{cases}$$

Coherent systems

A system is **coherent** provided:

Every loop in the interaction graph is positive.

Equivalently:

In each loop, there are no ambiguous edges and the number of negative edges is even.

Example The system

$$\begin{aligned} \dot{x} &= -x \\ \dot{y} &= y(1 + x + y - z) \\ \dot{z} &= z(\sin x - y - z) \end{aligned}$$

is coherent. It cannot be made cooperative by changing signs of variables. But replacing z by $-z$ makes the system quasimonotone.

Coherent systems can be made quasicooperative

Theorem 5. *A coherent system can be transformed to a quasicooperative system by permuting the variables and changing the signs of some of them.*
—Angelo, Hirsch & Sontag 2007

Goal: Adapt results on quasicooperative systems to coherent systems.

This seems to require restrictions on domains.

Attractors in coherent systems

A **coordinate halfspace** in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n \colon\ \alpha x_j \geq c\}$ for a single $j \in \{1, \ldots, n\}$, and $\alpha \in \{\pm 1\}$.

Theorem 6. *For coherent systems in $X \subset \mathbb{R}^n$ the following hold:*

- *Orbits are nowhere dense.*
- *Assume: X is open in $\mathbb{R}^n$ or a coordinate halfspace, $A \subset X$ is a connected attractor. Then no orbit is dense in A. In particular, there does not exist an attracting cycle.*
- *Assume: X is open in $\mathbb{R}^n$, there exists a global attractor, $\mathcal{E} = q$. Then q is globally asymptotically stable.*

Does this hold for more general domains X?
Conjecture: Yes, for convex X.

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